Radiative transfer implies a modified reciprocity relation

Raphael Aronson

Bioimaging Sciences Corporation, 64 Burnett Terrace, West Orange, New Jersey 07052

Received July 2, 1996; accepted September 13, 1996

The usual global reciprocity relations of radiative transfer do not hold for two points located in regions of different index of refraction. Modified reciprocity relations that involve the relative index are derived. The result has computational as well as theoretical consequences. © 1997 Optical Society of America
[S0740-3232(97)01202-7]

Key words: radiative transfer, transport theory, reciprocity, diffusion theory.

PACS numbers: 05.60.+w

1. INTRODUCTION

A common article of wisdom both in linear particle transport and in diffusion theory is that global reciprocity holds. That is, if source and detector are interchanged (and, in transport theory, if particle motions are reversed), the new detector readings are the same as the old ones, with no assumptions about the spatial arrangement of material. This paper will demonstrate that this reciprocity property fails for light when source and detector sit in regions of different index of refraction. Instead, it must be replaced by a modified reciprocity relation. There are requirements on the scattering, such as that detailed balance hold. Here we consider the case of elastic scattering, in the sense used by astrophysicists, in which the recoil energy of the scatterers is negligible. The discussion here will be restricted to time-independent problems.

The major applications of linear particle transport (and of diffusion, which is an approximation to linear transport) have been in neutron transport and in radiative transfer. In neutron transport the difficulty does not arise; in radiative transfer it evidently has not been considered before now. The motivation for the present work is the burgeoning application to medical imaging by light.[1] In the regime of interest in most of that work, one deals with monoenergetic photons (that is, Newtonian particles) subject to random collisions but obeying the laws of geometrical optics. Thus at discontinuities in the index of refraction, the reflection is specular and the refraction obeys Snell's law. A tool in frequent use is perturbation theory, which requires for its application solution of an adjoint problem.[2] The results of this paper are relevant to the usual procedure of obtaining the adjoint solution by solving a direct problem and using reciprocity.

The reciprocity relation can be stated as a symmetry property of the Green's function. In diffusion theory, this is[3]

$$G(\mathbf{r},\ \mathbf{r}') = G(\mathbf{r}',\ \mathbf{r}), \tag{1}$$

where $G(\mathbf{r}, \mathbf{r}')$ is the intensity at $\mathbf{r}$ resulting from a unit isotropic source at $\mathbf{r}'$. In transport theory it is[4]

$$G(r,\ \hat{\Omega};\ \mathbf{r}',\ \hat{\Omega}') = G(\mathbf{r}',\ -\Omega';\ \mathbf{r},\ -\hat{\Omega}), \tag{2}$$

where $G(\mathbf{r}, \hat{\Omega}; \mathbf{r}', \hat{\Omega}')$ is the angular intensity at $\mathbf{r}$ in direction $\hat{\Omega}$ due to a unit monodirectional source at $\mathbf{r}'$ in direction $\hat{\Omega}'$. That is, interchanging source and detector and simultaneously reversing particle directions does not change the detector reading. This reciprocity is a global consequence of time-reversal invariance.

Equation (1) arises from the fact that the diffusion problem is self-adjoint. It is well known that the Green's functions of self-adjoint operators are Hermitian. The transport operator is not self-adjoint. In general, there need be no reciprocity relation for the Green's function of a non-self-adjoint operator, as in the case, for instance, of the transport equation for neutrons in the slowing-down region. When the scattering kernel is symmetric, as in the situation discussed here, or when it can be symmetrized, as, for instance, for thermal neutrons, one can still obtain a reciprocity relation for the transport Green's function.

In the presence of nonuniform index of refraction the diffusion operator is no longer self-adjoint and Eq. (1) fails. Likewise, Eq. (2) fails in transport theory. The finding of this paper is that nevertheless, one can find generalized reciprocity relations, in which Eqs. (1) and (2) must be replaced respectively by the relations

$$G(\mathbf{r}',\ \mathbf{r}) = n^2 G(\mathbf{r},\ \mathbf{r}') \tag{3}$$

and

$$G(\mathbf{r}',\ \hat{\Omega};\ \mathbf{r},\ \hat{\Omega}) = n^2 G(\mathbf{r},\ -\hat{\Omega};\ \mathbf{r}',\ -\hat{\Omega}'), \tag{4}$$

where n is the relative index of refraction from $\mathbf{r}$ to $\mathbf{r}'$. It is not G but $G/n^2(\mathbf{r})$ that is symmetric, where $n(\mathbf{r})$ is the index at the field point.

Equation (1) follows from Eq. (2), as does Eq. (3) from Eq. (4), by averaging the transport theory results over initial direction and integrating over final direction. It must be proved separately for diffusion theory, however, because both the diffusion equation and the diffusion interface conditions are approximations to the transport results.

First diffusion theory and then transport theory is considered for two-region systems. The results are then extended to arbitrary variations in the index of refraction. The result is illustrated with a discussion of collisionless transport in two half-spaces, which allows an explicit solution. Finally, computational consequences of using the generalized reciprocity relations are discussed.

2. DIFFUSION THEORY

First consider diffusion theory because it is simpler. The diffusion Green's function satisfies the equation

$$\nabla \cdot D(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}') + \Sigma_a(\mathbf{r})G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (5)$$

in some volume V, where D is the diffusion coefficient and Σ_a is the macroscopic absorption cross section at $\mathbf{r}$. The usual boundary condition, whether the boundary is free or whether Fresnel reflection holds, is that G satisfies a linear boundary condition when $\mathbf{r}$ is on the surface S. The standard proof of reciprocity is to multiply Eq. (5) by $G(\mathbf{r}, \mathbf{r}'')$, subtract from it the corresponding relation with $\mathbf{r}'$ and $\mathbf{r}''$ interchanged, integrate over the volume, and use Green's theorem to change the term with the divergence into a surface integral. The result is

$$\int_S \mathrm{d}S\hat{\mathbf{e}} \cdot D(\mathbf{r})[G(\mathbf{r}, \mathbf{r}'')\nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla G(\mathbf{r}, \mathbf{r}'')] = G(\mathbf{r}', \mathbf{r}'') - G(\mathbf{r}'', \mathbf{r}'). \quad (6)$$

The unit vector in the outward normal direction is written here as $\hat{\mathbf{e}}$ to avoid confusion with the index of refraction n. The left-hand side vanishes for homogeneous linear boundary conditions of the form $aG + b\partial G/\partial e = 0$, where $\partial/\partial e$ is a normal derivative, and Eq. (1) follows. The self-adjointness of the diffusion operator has been invoked here explicitly to prove the result.

Now suppose that V is composed of two regions, V_1 and V_2, with index of refraction n_1 and n_2, respectively. The procedure in the last paragraph must be done separately for V_1 and V_2 because the Green's function is discontinuous on the interface S'. The surface integrals still vanish on the exterior portion of S but not on S'. For V_i, Eq. (6) must be replaced by

$$I_i \equiv \int_{S'} \mathrm{d}S\hat{\mathbf{e}}_i \cdot D(\mathbf{r}_i)[G(\mathbf{r}_i, \mathbf{r}'')\nabla G(\mathbf{r}_i, \mathbf{r}') - G(\mathbf{r}_i, \mathbf{r}')\nabla G(\mathbf{r}_i, \mathbf{r}'')] = H_i(\mathbf{r}')G(\mathbf{r}', \mathbf{r}'') - H_i(\mathbf{r}'')G(\mathbf{r}'', \mathbf{r}'), \quad (7)$$

where I_i designates the surface integral evaluated on the V_i side of S' and $\hat{\mathbf{e}}_i$ is the unit normal pointing outward from V_i (so that $\hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_2$). The symbol $\mathbf{r}_i$ in the arguments explicitly designates a surface point with the function defined on the V_i side. The symbol $H_i(\mathbf{r})$ represents the step function for V_i: It is unity if $\mathbf{r} \in V_i$ and zero otherwise. While the diffusion operator in each region is self-adjoint, the problem is not; the discontinuity in the Green's function spoils the self-adjointness, which would require that $I_1 = -I_2$.

The flux due to a unit source at $\mathbf{r}'$ is given by $\mathbf{J}(\mathbf{r}, \mathbf{r}') = -D(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}')$ and must be continuous everywhere, including interfaces. For particles, the second interface condition is that G be continuous. For photons obeying geometrical optics, the flux is still continuous, since this is just a statement of energy conservation. The appropriate second interface condition here is[5]

$$n^2G(\mathbf{r}_1, \mathbf{r}') - G(\mathbf{r}_2, \mathbf{r}') = C(n)\hat{\mathbf{e}}_1 \cdot \mathbf{J}(\mathbf{r}, \mathbf{r}'), \quad (8)$$

where n is the relative index of refraction from V_1 to V_2 and $C(n)$ is a constant depending on n alone. The symbol $\mathbf{r}$ rather than $\mathbf{r}_1$ or $\mathbf{r}_2$ appears on the right-hand side of Eq. (8) because the expression is continuous. This relation was derived by a generalization of the simple kinetic theory argument that leads to Fick's law. Then

$$\int_{S'} \mathrm{d}S\hat{\mathbf{e}}_2 \cdot D(\mathbf{r}_2)G(\mathbf{r}_2, \mathbf{r}'')\nabla G(\mathbf{r}_2, \mathbf{r}') = \int_{S'} \mathrm{d}S\hat{\mathbf{e}}_1 \cdot J(\mathbf{r}, \mathbf{r}')[n^2G(\mathbf{r}_1, \mathbf{r}'') - C(n)\hat{\mathbf{e}}_1 \cdot J(\mathbf{r}, \mathbf{r}'')]. \quad (9)$$

It follows that $I_2 = -n^2I_1$, and so

$$[n^2H_1(\mathbf{r}') + H_2(\mathbf{r}')]G(\mathbf{r}', \mathbf{r}'') = [n^2H_1(\mathbf{r}'') + H_2(\mathbf{r}'')]G(\mathbf{r}'', \mathbf{r}'). \quad (10)$$

Equation (3) follows immediately.

3. TRANSPORT THEORY

The linear transport equation for the Green's function is

$$\hat{\Omega} \cdot \nabla G(\mathbf{r}, \hat{\Omega}; \mathbf{r}', \hat{\Omega}') + \Sigma_t(\mathbf{r})G(\mathbf{r}, \hat{\Omega}; \mathbf{r}', \hat{\Omega}') = \int \mathrm{d}\Omega_1\Sigma_s(\mathbf{r}; \hat{\Omega}_1 \rightarrow \hat{\Omega})G(\mathbf{r}, \hat{\Omega}_1; \mathbf{r}', \hat{\Omega}') + \delta(\mathbf{r} - \mathbf{r}')\delta(\hat{\Omega} - \hat{\Omega}'). \quad (11)$$

In the usual proof of reciprocity, one multiplies this equation by $G''(\mathbf{r}, -\hat{\Omega}) \equiv G(\mathbf{r}, -\hat{\Omega}; \mathbf{r}'', -\hat{\Omega}'')$, subtracts from it $G'(\mathbf{r}, \hat{\Omega}) \equiv G(\mathbf{r}, \hat{\Omega}; \mathbf{r}', \hat{\Omega}')$ multiplied by the equation for $G''(\mathbf{r}, -\hat{\Omega})$, and integrates over all $\mathbf{r}$ and $\hat{\Omega}$. The second term on the left makes no contribution. Neither does the scattering term, by virtue of the reciprocity of the differential scattering cross section required by time-reversal invariance, to wit: $\Sigma_s(\mathbf{r}; \hat{\Omega}_1 \rightarrow \hat{\Omega}) = \Sigma_s(\mathbf{r}; -\hat{\Omega} \rightarrow -\hat{\Omega}_1)$. The result is

$$\int_S \mathrm{d}\mathbf{r} \int \mathrm{d}\hat{\Omega}\hat{\mathbf{e}} \cdot \hat{\Omega}G''(\mathbf{r}, -\hat{\Omega})G'(\mathbf{r}, \hat{\Omega}) = G''(\mathbf{r}', -\hat{\Omega}') - G'(\mathbf{r}'', \hat{\Omega}''). \quad (12)$$

The left-hand side vanishes for both a free surface (no radiation incident from the outside) and for specular reflection. In the former case, the boundary condition is $G(\mathbf{r}, \hat{\Omega}) = 0$ when $\mathbf{r} \in S$ and $\hat{\mathbf{e}} \cdot \hat{\Omega} < 0$, where the generic G is introduced to represent either G' or G'' and the argument $\mathbf{r}$ in G is suppressed for surface points. Thus either $G''(\mathbf{r}, -\hat{\Omega})$ or $G'(\mathbf{r}, \hat{\Omega})$ vanishes on the surface for all $\hat{\Omega}$. For specular reflection,

$$G(-\hat{\Omega}) = r(\hat{\mathbf{e}} \cdot \hat{\Omega})G(\hat{\Omega}_r), \qquad \mathbf{r} \in S, \qquad \hat{\mathbf{e}} \cdot \hat{\Omega} > 0, \quad (13)$$

where $-\hat{\Omega}_r$ is the direction of the reflected ray for incident direction $\hat{\Omega}$, and $r(\hat{\mathbf{e}}\cdot\hat{\Omega})$ is the reflection probability. The surface integral in Eq. (12) can thus be rewritten as

$$\int_{\hat{\mathbf{e}}\cdot\hat{\Omega}>0} \mathrm{d}\hat{\Omega}\hat{\mathbf{e}}\cdot\hat{\Omega} r(\hat{\mathbf{e}}\cdot\hat{\Omega})[G''(\hat{\Omega}_r)G'(\hat{\Omega}) - G''(\hat{\Omega})G'(\hat{\Omega}_r)]. \tag{14}$$

Since $\hat{\mathbf{e}}\cdot\hat{\Omega} = \hat{\mathbf{e}}\cdot\hat{\Omega}_r$ and $\hat{\Omega}$ and $\hat{\Omega}_r$ are 180° apart in azimuth about $\hat{\mathbf{e}}$, the integral vanishes.

For the two-region problem, again there is a surface contribution only from S', so

$$\begin{aligned} I_i &\equiv \int_{S'} \mathrm{d}\mathbf{r} J_i(\mathbf{r}, \mathbf{r}', \hat{\Omega}', \mathbf{r}'', \hat{\Omega}'') \\ &= H_i(\mathbf{r}')G''(\mathbf{r}', -\hat{\Omega}') - H_i(\mathbf{r}'')G'(\mathbf{r}'', \hat{\Omega}''), \end{aligned} \tag{15}$$

where

$$J_i \equiv \int \mathrm{d}\hat{\Omega}_i\hat{\mathbf{e}}_i\cdot\hat{\Omega}_i G''(-\hat{\Omega}_i)G'(\hat{\Omega}_i). \tag{16}$$

The arguments on J_i have been suppressed here.

While reflection contributes nothing to these integrals, transmission does. Conservation of energy gives the interface condition

$$\mathrm{d}\hat{\Omega}_j\hat{\mathbf{e}}_j\cdot\hat{\Omega}_j G(\hat{\Omega}_j) = -\mathrm{d}\hat{\Omega}_i\hat{\mathbf{e}}_i\cdot\hat{\Omega}_i t_{i\to j}(\hat{\mathbf{e}}_i\cdot\hat{\Omega}_i)G(\hat{\Omega}_i),$$

$$\hat{\mathbf{e}}_i\cdot\hat{\Omega}_i > 0. \tag{17}$$

Here $\hat{\Omega}_i$ is the direction vector of a ray on the V_i side of the interface, $\hat{\Omega}_j$ is the direction vector of the same ray (refracted) on the V_j side [so that $G(\hat{\Omega}_i)$ is to be evaluated on the V_i-side of the interface], and $t_{i\to j}(\hat{\mathbf{e}}_i\cdot\hat{\Omega}_i)$ is the fraction of the energy incident on the interface at $\mathbf{r}$ in direction $\hat{\Omega}_i$ transmitted from V_i to V_j. The minus sign comes from the fact that $\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1$ and $\hat{\mathbf{e}}_2\cdot\hat{\Omega}_2$ have opposite signs, since $\hat{\mathbf{e}}_1 = -\hat{\mathbf{e}}_2$. Local reciprocity demands that

$$t_{i\to j}(\hat{\mathbf{e}}_i\cdot\hat{\Omega}_i) = t_{j\to i}(-\hat{\mathbf{e}}_j\cdot\hat{\Omega}_j). \tag{18}$$

The Fresnel formulas explicitly obey this reciprocity.[6] From Snell's law one derives that

$$\mathrm{d}\hat{\Omega}_1\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1 = -n^2\mathrm{d}\hat{\Omega}_2\hat{\mathbf{e}}_2\cdot\hat{\Omega}_2. \tag{19}$$

This, together with Eq. (17), gives the n^2 law of radiance, $G(\hat{\Omega}_2) = n^2 t_{1\to 2}(\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1)G(\hat{\Omega}_1)$, where $\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1 > 0$. It follows that

$$\begin{aligned} J_2 &= \int_{\hat{e}_1\cdot\hat{\Omega}_1>0} \mathrm{d}\hat{\Omega}_2\hat{\mathbf{e}}_2\cdot\hat{\Omega}_2[G''(-\hat{\Omega}_2)G'(\hat{\Omega}_2) \\ &\quad - G''(\hat{\Omega}_2)G'(-\hat{\Omega}_2)] \\ &= -\int_{\hat{e}_1\cdot\hat{\Omega}_1>0} \mathrm{d}\hat{\Omega}_1\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1 t_{1\to 2}(\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1) \\ &\quad \times [G''(-\hat{\Omega}_2)G'(\hat{\Omega}_1) - G''(\hat{\Omega}_1)G'(-\hat{\Omega}_2)] \\ &= -\int_{\hat{e}_1\cdot\hat{\Omega}_1>0} \mathrm{d}\hat{\Omega}_1\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1 t_{2\to 1}(-\hat{\mathbf{e}}_2\cdot\hat{\Omega}_2) \\ &\quad \times [G''(-\hat{\Omega}_2)G'(\hat{\Omega}_1) - G''(\hat{\Omega}_1)G'(-\hat{\Omega}_2)] \\ &= \int_{\hat{e}_1\cdot\hat{\Omega}_1<0} \mathrm{d}\hat{\Omega}_1\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1 t_{2\to 1}(\hat{\mathbf{e}}_2\cdot\hat{\Omega}_2) \\ &\quad \times [G''(\hat{\Omega}_2)G'(-\hat{\Omega}_1) - G''(-\hat{\Omega}_1)G'(\hat{\Omega}_2)] \\ &= -n^2\int_{\hat{e}_1\cdot\hat{\Omega}_1<0} \mathrm{d}\hat{\Omega}_2\hat{\mathbf{e}}_2\cdot\hat{\Omega}_2 t_{2\to 1}(\hat{\mathbf{e}}_2\cdot\hat{\Omega}_2) \\ &\quad \times [G''(\hat{\Omega}_2)G'(-\hat{\Omega}_1) - G''(-\hat{\Omega}_1)G'(\hat{\Omega}_2)] \\ &= n^2\int_{\hat{e}_1\cdot\Omega_1<0} \mathrm{d}\hat{\Omega}_1\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1[G''(\hat{\Omega}_1)G'(-\hat{\Omega}_1) \\ &\quad - G''(-\hat{\Omega}_1)G'(\hat{\Omega}_1)] \\ &= -n^2\int \mathrm{d}\hat{\Omega}_1\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1 G''(-\hat{\Omega}_1)G'(\hat{\Omega}_1) = -n^2 J_1. \end{aligned} \tag{20}$$

The first equality here comes from writing the integral in Eq. (16) in terms of directions such that $\hat{\mathbf{e}}_1\cdot\hat{\Omega}_1 > 0$; the second uses Eq. (17); the third, Eq. (18); the fourth replaces $\hat{\Omega}$ with $-\hat{\Omega}$; the fifth uses Eq. (19); the sixth, Eq. (17) again; and the seventh equality extends the integral to all $\hat{\Omega}$. It follows from Eq. (20) that $n^2 I_1 + I_2 = 0$ and thus that

$$\begin{aligned} &[n^2 H_1(\mathbf{r}') + H_2(\mathbf{r}')]G''(\mathbf{r}', -\hat{\Omega}') \\ &\quad = [n^2 H_1(\mathbf{r}'') + H_2(\mathbf{r}'')]G'(\mathbf{r}'', \hat{\Omega}''), \end{aligned} \tag{21}$$

which gives Eq. (4).

A. Generalization to General Geometries

To extend the diffusion theory results beyond a simple two-region geometry, consider any closed surface S with local outward normal $\hat{\mathbf{e}}$ and define

$$\begin{aligned} K &\equiv \int_S \mathrm{d}S\hat{\mathbf{e}}\cdot D(r)[G(\mathbf{r}, \mathbf{r}'')\nabla G(\mathbf{r}, \mathbf{r}') \\ &\quad - G(\mathbf{r}, \mathbf{r}')\nabla G(\mathbf{r}, \mathbf{r}'')]/n^2(\mathbf{r}), \end{aligned} \tag{22}$$

where $n(\mathbf{r})$ is the local index of refraction at $\mathbf{r}$. It follows from the results above that while $n(\mathbf{r})$ may not be continuous, the surface integral K is invariant for surfaces S that can be deformed into one another by a continuous transformation without passing through $\mathbf{r}'$ or $\mathbf{r}''$. Let S enclose both points. It was shown above that when S is taken as the outer boundary of the system, $K = 0$. Deform the surface to two infinitesimal spheres, one surrounding $\mathbf{r}'$ and one surrounding $\mathbf{r}''$. On the first sphere, $n(\mathbf{r}) = n(\mathbf{r}')$ and on the second, $n(\mathbf{r}) = n(\mathbf{r}'')$, while $K = 0$. Thus

$$G(\mathbf{r}', \mathbf{r}'')/n^2(\mathbf{r}') - G(\mathbf{r}'', \mathbf{r}')/n^2(\mathbf{r}'') = 0. \tag{23}$$

Since the relative index of refraction is $n = n(\mathbf{r}'')/n(\mathbf{r}')$, Eq. (3) follows. The same argument leads to Eq. (4) in transport theory, where here one defines

$$K = \int_S \mathrm{d}\mathbf{r}/n^2(\mathbf{r}) \int \mathrm{d}\hat{\Omega}\hat{\mathbf{e}} \cdot \hat{\Omega} G''(\mathbf{r}, -\hat{\Omega}) G'(\mathbf{r}, \hat{\Omega}). \tag{24}$$

B. Example: Collisionless Transport for Two Half-spaces

The results of this paper are hardly intuitive. One can perhaps see better how the modified reciprocity comes about by examining the result of explicit calculation in a simple case, for example, for purely absorbing media in plane geometry.

Consider two uniform nonscattering half-spaces with interface at $x = 0$. For $x > 0$ the index of refraction is unity and the macroscopic absorption cross section is Σ; for $x < 0$ the index is n and the cross section Σ'. A plane source at $x' < 0$ emits 1 photon/s into a small solid angle $\Delta\hat{\Omega}'$ making angle $\cos^{-1}\mu'$ with the positive x axis, and a collimated detector at $x > 0$ accepts photons in a small solid angle $\Delta\hat{\Omega}$ centered about the direction given by $\cos^{-1}\mu$. Let us examine first the case of no absorption. If the photons from the source do not enter the detector, the Green's function is zero, and clearly this is also the case if source and detector are interchanged with the same collimation directions, so reciprocity holds trivially. The other situation is that $\Delta\hat{\Omega}$ is the solid angle into which the emitted beam is refracted. In that case, the flux in the interval $(x', 0)$, and thus at the flux incident on the interface, is μ', so the transmitted flux is $\mu' t$, where t is the transmission probability for photons with angle of incidence $\cos^{-1}\mu'$ from the left. The angular intensity at the detector is thus $\mu' t/\Delta\hat{\Omega}$. Absorption puts in some exponential factors. The end result is that

$$G(x, \hat{\Omega}; x', \hat{\Omega}') = (\mu' t/\Delta\hat{\Omega})\exp(-\Sigma'|x'|/\mu' - \Sigma x/\mu). \tag{25}$$

The same argument, interchanging source and detector, gives

$$G(x', -\hat{\Omega}'; x, -\hat{\Omega}) = (\mu t/\Delta\hat{\Omega}')\exp(-\Sigma'|x'|/\mu' - \Sigma x/\mu), \tag{26}$$

since t is symmetric. But to first order in $\Delta\hat{\Omega}$, Snell's law says that $\mu\Delta\hat{\Omega} = n^2\mu'\Delta\hat{\Omega}'$, so

$$G(x', -\hat{\Omega}'; x, -\hat{\Omega}) = n^2 G(x, \hat{\Omega}; x', \hat{\Omega}'), \tag{27}$$

which is a version of Eq. (4).

4. DISCUSSION

It has been shown that it is necessary to modify the usual reciprocity relations for the time-independent diffusion and transport equations according to both Eqs. (3) and (4) when the index of refraction at the field point is different from that at the source point.

The approach in this paper has been to derive the results for a continuous change in the index by going to the limit from the results for a discontinuous change. Pomraning[7] derived a transport equation for photons in the presence of a continuous index. His derivation is very formal, though it is possible to derive his equation, at least in the one-wavelength picture used here, in a much more physical way. I have proved Eq. (4) also from his equations, but the procedure is no simpler than the derivation given here.

Besides being of theoretical interest, the necessary modification of the customary reciprocity relations has practical computational consequences. In the application that led to this investigation, the transport of light in tissue, the problem of interest is an inverse problem.[1] One wants to know the collision cross sections locally within the tissue, given the scattered intensity distribution at the surface of the body or some subset of that information. Much work has been done using a perturbation approach, for which the decrease in the intensity at a detector position $\mathbf{r}_d$ of a given change $\delta\Sigma_a(\mathbf{r})$ in the absorption cross section at $\mathbf{r}$, for instance, is given in diffusion theory by $\int \delta\Sigma_a(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_s) G(\mathbf{r}_d, \mathbf{r}) \mathrm{d}\mathbf{r}$, where $\mathbf{r}_s$ is the source position. The validity of this expression can be seen by a physical argument. The quantity $\delta\Sigma_a(\mathbf{r}) \times G(\mathbf{r}, \mathbf{r}_s)$ is the change in the absorption density at $\mathbf{r}$ that is due to $\delta\Sigma_a(\mathbf{r})$, which serves as a negative volume source for diffusion from $\mathbf{r}$ to $\mathbf{r}_d$. The Green's function $G(\mathbf{r}_d, \mathbf{r})$ is usually computed by taking the *detector* as a source and $\mathbf{r}$ as the field point and using the reciprocity relation. In most medical imaging applications the detector is in air and $\mathbf{r}$ is in tissue. For a typical index of refraction for tissue of approximately 1.4, the result of using Eq. (1) or Eq. (2) would be equivalent to making an error of a factor of 2 in all the detector readings. For changes in other cross sections, such as scattering, or for other detectors (surface flux detectors, for instance), the result is the same, as it is if transport theory is used instead.

ACKNOWLEDGMENTS

Thanks are due to Gerald Pomraning for calling to the author's attention his work on radiative transfer in a medium with a continuous index of refraction. This work was supported in part by the National Institutes of Health grant RO1 CA59955 and by Office of Naval Research grant N000149510063.

The author can be reached at tel, 201-325-3342; fax, 201-325-1089; e-mail, Rlaron@aol.com.

REFERENCES

1. See, for instance, R. L. Barbour, H. L. Graber, Y. Wang, J. Chang, and R. Aronson, "A perturbation approach for optical diffusion tomography using continuous-wave and time-resolved data," in *Medical Optical Tomography: Functional Imaging and Monitoring*, G. Müller *et al.*, eds., Vol. IS11 of Institute Series (SPIE, Bellingham, Wash., 1993),

pp. 87–120; S. R. Arridge, "The forward and inverse problems in time-resolved infra-red imaging," pp. 35–64.
2. J. J. Duderstadt and W. R. Martin, *Transport Theory* (Wiley, New York, 1979), pp. 369–373.
3. J. R. Lamarsh, *Nuclear Reactor Theory* (Addison-Wesley, Reading, Mass., 1966), Eq. (5–126), p. 156.
4. K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Addison-Wesley, Reading, Mass. 1967), Eq. (9), p. 27.
5. R. Aronson, "Boundary conditions for diffusion of light," J. Opt. Soc. Am. A **12**, 2532–2539 (1995).
6. M. Born and E. Wolf, *Principles of Optics*, 2nd rev. ed. (Pergamon, New York, 1964).
7. G. Pomraning, *The Equations of Radiation Hydrodynamics* (Pergamon, Oxford, 1973), Chap. 5, pp. 144–153. The final result is contained in Eq. (5.67).